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Two-Parameter Deformed Multimode Usui Operators of Multimode $SU(2)_{q,s}$ and $SU(1, 1)_{q,s}$ Algebras

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The two-parameter deformed (q,s-deformed) multimode Usui operators of multimode $SU(2)_{q,s}$ and $SU(1, 1)_{q,s}$ algebras are introduced and the multimode q, s-Dyson boson realizations of both algebras are also given.

1. INTRODUCTION

The Usui operator is very useful in studying the transformation of fermion states into boson states (Usui, 1960). Recently there has been growing interest in the quantum algebras, in particular their q-boson realizations (Chaichian *et al.*, 1990; Fu, 1990; Kulish and Damaskinsky, 1990; Solomon and Katriel, 1990) and their single-mode q-Usui operators (Yu, 1992). The present paper constructs the multimode q,s-Usui operators of multimode $SU(2)_{q,s}$ and $SU(1, 1)_{q,s}$ algebras.

2. THE GENERAL USUI OPERATOR

The general Usui operator is defined as follows (Usui, 1960):

$$|\tilde{\psi}\rangle = U|\psi\rangle = \{\langle\Lambda|\exp(\sum_{\alpha\cdot\Lambda>0} b_{\alpha}^{+} E_{\alpha})|0\rangle\}|\psi\rangle \tag{1}$$

where the Lie algebra $\mathscr{A}$ is assumed to be in the standard form

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$$\mathscr{A} = \{H_i, E_{\pm\alpha} | i = 1, 2, \ldots, l; \alpha \in \text{positive roots of } \mathscr{A}\} \tag{2}$$

and $|\Lambda\rangle$ is a highest weight state in the irreducible representative space V, with the following properties:

$$H_i|\Lambda\rangle = \Lambda_i|\Lambda\rangle \qquad (i = 1, 2, \ldots, l) \tag{3}$$

$$E_\alpha|\Lambda\rangle = 0 \qquad (\alpha \cdot \Lambda \geq 0) \tag{4}$$

$$E_\alpha|\Lambda\rangle = 0 \qquad (\alpha \cdot \Lambda < 0) \tag{5}$$

According to Eq. (1), we have the boson image of an arbitrary operator $\hat{O}$,

$$|\phi\rangle = \hat{O}|\psi\rangle \rightarrow |\tilde{\phi}\rangle = U\hat{O}|\psi\rangle = \hat{O}^{(\mathrm{D})}|\tilde{\psi}\rangle \tag{6}$$

where the operator $O^{(\mathrm{D})}$ is called the generalized Dyson realization for operator $\hat{O}$.

3. THE MULTIMODE *q,s*-USUI OPERATOR AND MULTIMODE *q,s*- DYSON REALIZATION OF MULTIMODE $SU(2)_{q,s}$ ALGEBRA

The multimode q,s-harmonic oscillators $\{A_k^+, A_k, N_k^a\}$ and $\{B_k^+, B_k, N_k^b\}$ satisfy the commutative relations (Yu and Liu, 1998)

$$A_k A_k^+ - s^{-1}qA_k^+A_k = (sq)^{-N_k^a}, \qquad A_kA_k^+ - (sq)^{-1}A_k^+A_k = (s^{-1}q)^{N_k^a} \tag{7}$$

$$[N_k^a, A_k] = -A_k, \qquad [N_k^a, A_k^+] = A_k^+ \tag{8}$$

$$B_kB_k^+ - sqB_k^+B_k = (sq^{-1})^{N_k^b} \tag{9}$$

$$[N_k^b, B_k] = -B_k, \qquad [N_k^b, B_k^+] = B_k^+ \tag{10}$$

where

$$A_k = a_1a_2 \ldots a_k \left\{\frac{[n_1^a]_{q,s}[n_2^a]_{q,s} \ldots [n_k^a]_{q,s}}{\min([n_1^a]_{q,s}, [n_2^a]_{q,s} \ldots, [n_k^a]_{q,s})}\right\}^{-1/2} \tag{11}$$

$$B_k = b_1b_2 \ldots b_k \left\{\frac{[n_1^b]_{q,s^{-1}}[n_2^b]_{q,s^{-1}} \ldots [n_k^b]_{q,s^{-1}}}{\min([n_1^b]_{q,s^{-1}}, [n_2^b]_{q,s^{-1}}, \ldots, [n_k^b]_{q,s^{-1}})}\right\}^{-1/2} \tag{12}$$

Here

$$N_k^a = \min(n_1^a, n_2^a, \ldots, n_k^a), \qquad N_k^b = \min(n_1^b, n_2^b, \ldots, n_k^b) \tag{13}$$

$$[x]_{q,s} = s^{1-x}[x] = s^{1-x}(q^x - q^{-x})/(q - q^{-1}), \qquad [x]_{q,s^{-1}} = s^{x-1}[x] \tag{14}$$

$$A_k^+A_k = [N_k^a]_{q,s}, \qquad A_kA_k^+ = [N_k^a + 1]_{q,s} \tag{15}$$

$$B_k^+ B_k = [N_k^b]_{q,s^{-1}}, \qquad B_k B_k^+ = [N_k^b + 1]_{q,s^{-1}} \tag{16}$$

The generators of the multimode $SU(2)_{q,s}$ algebra can be obtained from its Jordan–Schwinger realization (Yu and Liu, 1998)

$$J_k^+ = A_k^+ B_k, \qquad J_k^- = B_k^+ A_k, \qquad J_k^0 = \frac{1}{2}(N_k^a - N_k^b) \tag{17}$$

which obey the commutation relations

$$[J_k^0, J_k^\pm] = \pm J_k^\pm, \qquad s^{-1} J_k^+ J_k^- - s J_k^- J_k^+ = s^{-2J_k^0}[2J_k^0] \tag{18}$$

If the parameters q and s are not roots of unity, then the basis vectors in the space V are

$$\begin{aligned} &|j, m; j, m; \ldots\rangle \\ &\quad = |j + m; j + m; \ldots\rangle \otimes |j - m; j - m; \ldots\rangle \\ &\quad = \frac{(A_k^+)^{j+m}}{\sqrt{[j+m]_{q,s}!}} |0, 0 \ldots\rangle \otimes \frac{(B_k^+)^{j-m}}{\sqrt{[j-m]_{q,s^{-1}}!}} |0, 0, \ldots\rangle \\ &(-j \le m \le j) \end{aligned} \tag{19}$$

We now define the multimode q,s-Usui operator of the multimode $SU(2)_{q,s}$ algebra as follows:

$$U = \langle j, -j; j, -j; \ldots| \exp_{q,s}(A_k^+ J_k^-)|0\rangle \tag{20}$$

where $\exp_{q,s}(x)$ is the q,s-exponential function

$$\exp_{q,s}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q,s}!} \tag{21}$$

So we have the multimode q,s-Dyson images of J_k^0 and $J_k^\pm$

$$(J_k^+)^{(\mathrm{D})} = A_k^+[2j - N_k^a]_{q,s^{-1}}, \qquad (J_k^-)^{(\mathrm{D})} = A_k, \qquad (J_k^0)^{(\mathrm{D})} = -j + N_k^a \tag{22}$$

Utilizing the Baker–Campbell–Hausdoff formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots \tag{23}$$

we have

$$A_k\, q^{N_k^a} = q q^{N_k^a} A_k, \qquad A_k q^{-N_k^a} = q^{-1}\, q^{-N_k^a} A_k \tag{24}$$

Equation (24) also holds for q replaced by s. According to Eq. (24), we can prove Eq. (22) holds Eq. (18).

4. THE MULTIMODE *q,s*-USUI OPERATOR AND MULTIMODE *q,s*-DYSON REALIZATION OF MULTIMODE $SU(1, 1)_{q,s}$

In order to construct the multimode $SU(1, 1)_{q,s}$ algebra, it is necessary to introduce two other independent multimode q,s-harmonic oscillators

$$C_k = c_1 c_2 \cdots c_k \left\{ \frac{[n_1^c]_{q,s}[n_2^c]_{q,s} \cdots [n_k^c]_{q,s}}{\min([n_1^c]_{q,s}, [n_2^c]_{q,s}, \ldots, [n_k^c]_{q,s})} \right\}^{-1/2} \tag{25}$$

$$D_k = d_1 d_2 \cdots d_k \left\{ \frac{[n_1^d]_{q,s^{-1}}[n_2^d]_{q,s^{-1}} \cdots [n_k^d]_{q,s^{-1}}}{\min([n_1^d]_{q,s^{-1}}, [n_2^d]_{q,s^{-1}}, \ldots, [n_k^d]_{q,s^{-1}})} \right\}^{-1/2} \tag{26}$$

where

$$c_i^+ c_i = [n_i^c]_{q,s}, \qquad c_i c_i^+ = [n_i^c + 1]_{q,s},$$

$$[n_i^c, c_i^+] = c_i^+, \qquad [n_i^c, c_i] = -c_i \tag{27}$$

$$c_i c_i^+ - s^{-1} q c_i^+ c_i = (sq)^{-n_i^c}, \qquad c_i c_i^+ - (sq)^{-1} c_i^+ c_i = (s^{-1}q)^{n_i^c} \tag{28}$$

$$d_j^+ d_j = [n_j^d]_{q,s^{-1}}, \qquad d_j d_j^+ = [n_j^d + 1]_{q,s^{-1}},$$

$$[n_j^d, d_j^+] = d_j^+, \qquad [n_j^d, d_j] = -d_j \tag{29}$$

$$d_j d_j^+ - sq d_j^+ d_j = (sq^{-1})^{n_j^d} \tag{30}$$

with $i, j = 1, 2, \ldots, k$. It is easy to check that

$$C_k C_k^+ - s^{-1} q C_k^+ C_k = (sq)^{-N_k^c}, \qquad C_k C_k^+ - (sq)^{-1} C_k^+ C_k = (s^{-1}q)^{N_k^c} \tag{31}$$

$$[N_k^c, C_k^+] = C_k^+, \qquad [N_k^c, C_k] = -C_k \tag{32}$$

$$D_k D_k^+ - sq D_k^+ D_k = (sq^{-1})^{N_k^d} \tag{33}$$

$$[N_k^d, D_k^+] = D_k^+, \qquad [N_k^d, D_k] = -D_k \tag{34}$$

with

$$N_k^c = \min(n_1^c, n_2^c, \ldots, n_k^c), \qquad N_k^d = \min(n_1^d, n_2^d, \ldots, n_k^d) \tag{35}$$

Similar to the single-mode $SU(1,1)_{q,s}$ algebra (Yu *et al.*, 1997a, b), the multimode $SU(1,1)_{q,m}$ algebra also has three types of unitary irreducible representations: a positive discrete series (a), a negative discrete series (b), and a continuous series, which we do not consider here. The generators of the multimode $SU(1,1)_{q,s}$ algebra can be obtained from a Jordan–Schwinger realization in terms of the above oscillators $\{A_k^+, A_k, N_k^a\}$, $\{B_k^+, B_k, N_k^b\}$, $\{C_k^+, C_k, N_k^c\}$, and $\{D_k^+, D_k, N_k^d\}$:

$$(J_k^{(a)})_+ = s^{-1} A_k^+ C_k^+, \qquad (J_k^{(a)})_- = s^{-1} C_k A_k \tag{36}$$

$$(J_k^{(a)})_0 = \frac{1}{2} (N_{k,a}^{(a)} + N_{k,c}^{(a)} + 1) \tag{37}$$

and

$$(J_k^{(b)})_+ = sB_k D_k, \qquad (J_k^{(b)})_- = SD_k^+ B_k^+ \tag{38}$$

$$(J_k^{(b)})_0 = \frac{-1}{2} (N_{k,b}^{(b)} + N_{k,d}^{(b)} + 1) \tag{39}$$

where

$$N_{k,a}^{(a)} = N_k^a, \qquad N_{k,c}^{(a)} = N_k^c, \qquad N_{k,b}^{(b)} = N_k^b, \qquad N_{k,d}^{(b)} = N_k^d \tag{40}$$

which obey the commutation relations

$$[(J_k^{(a)})_0, (J_k^{(a)})_\pm] = \pm(J_k^{(a)})_\pm \tag{41}$$

$$s^{-1}(J_k^{(a)})_+ (J_k^{(a)})_- - s(J_k^{(a)})_- (J_k^{(a)})_+ = -s^{-2(J_k^{(a)})_0}[2(J_k^{(a)})_0] \tag{42}$$

and

$$[(J_k^{(b)})_0, (J_k^{(b)})_\pm] = \pm(J_k^{(b)})_\pm \tag{43}$$

$$s^{-1}(J_k^{(b)})_+(J_k^{(b)})_- - s(J_k^{(b)})_-(J_k^{(b)})_+ = -s^{-2(J_k^{(b)})_0}[2(J_k^{(b)})_0] \tag{44}$$

If q and s are not roots of unity, the basis vectors for both the positive and negative discrete cases in the irreducible space V are, respectively,

$$|l, r; l, r; \ldots\rangle^a = |r - l - 1; r - l - 1; \ldots\rangle^a \otimes |r + l; r + l; \ldots\rangle^a \qquad (r \geq -l > 0) \tag{45}$$

$$|l, r; l, r; \ldots\rangle^b = |-r - l - 1; -r - l - 1; \ldots\rangle^b \otimes |-r + l; -r + l; \ldots\rangle^b \qquad (r \leq l < 0) \tag{46}$$

The actions of the generators of the multimode $SU(1, 1)_{q,s}$ on the basis vectors are, respectively,

$$(J_k^{(a)})_+ |l, r; l, r; \ldots\rangle^a = s^{-1} \sqrt{[r-l]_{q,s} [r + l + 1]_{q,s}}\; |l, r + 1; l, r + 1; \ldots\rangle^a \tag{47}$$

$$(J_k^{(a)})_- |l, r; l, r; \ldots\rangle^a = s^{-1} \sqrt{[r + l]_{q,s}[r - l - 1]_{q,s}}\; |l, r - 1; l, r - 1; \ldots\rangle^a \tag{48}$$

$$(J_k^{(a)})_0 |l, r; l, r; \ldots\rangle^a = r|l, r; l, r; \ldots\rangle^a \tag{49}$$

and

$$(J_k^{(b)})_+ \,|l, r; l, r; \ldots\rangle^b = s\sqrt{[-r-l-1]_{q,s^{-1}}[-r+l]_{q,s^{-1}}}\,|l, r+1; l, r+1; \ldots\rangle^b \tag{50}$$

$$(J_k^{(b)})_- \,|l, r; l, r; \ldots\rangle^b = s\sqrt{[-r-l]_{q,s^{-1}}[-r+l+1]_{q,s^{-1}}}\,|l, r-1; l, r-1; \ldots\rangle^b \tag{51}$$

$$(J_k^{(b)})_0 \,|l, r; l, r; \ldots\rangle^b = r|l, r; l, r; \ldots\rangle^b \tag{52}$$

The multimode q,s-Usui operators for the positive and negative discrete cases of the multimode $SU(1,1)_{q,s}$ algebra can be defined as

$$U^{(a)} = {}^a\langle l, -l; l, -l; \ldots |\exp_{q,s}(A_k^+(J_k^{(a)})_+)|0\rangle \tag{53}$$

$$U^{(b)} = {}^b\langle l, l; l, l; \ldots |\exp_{q,s^{-1}}(B_k^+(J_k^{(b)})_-)|0\rangle \tag{54}$$

The multimode q,s-Dyson images of generators of multimode $SU(1,1)_{q,s}$ are

$$(J_k^{(a)})_+^{(D)} = s^{-1}A_k^+[2l + N_k^a]_{q,s}, \qquad (J_k^{(a)})_-^{(D)} = s^{-1}A_k, \qquad (J_k^{(a)})_0^{(D)} = l + N_k^a \tag{55}$$

and

$$(J_k^{(b)})_+^{(D)} = sB_k, \qquad (J_k^{(b)})_-^{(D)} = sB_k^+[2l + N_k^b]_{q,s^{-1}}, \qquad (J_k^{(b)})_0^{(D)} = -(l + N_k^b) \tag{56}$$

It is easy to check that Eq. (55) holds Eqs. (41) and (42) as well as Eq. (56) holds Eqs. (43) and (44), respectively.

From the above discussion, we see that the multimode q,s-Dyson realizations for multimode $SU(2)_{q,s}$ and $SU(1,1)_{q,s}$ algebras are partly nonunitary. In order to overcome the difficult of nonunitary, similar to Yu (1992), we can obtain new unitary realizations, Holstein–Primakoff realizations, as follows:

(1) The multimode $SU(2)_{q,s}$ algebra

$$\begin{aligned}(J_k^+)^{(HP)} &= A_k^+\sqrt{[2j - N_k^a]_{q,s^{-1}}},\\ (J_k^-)^{(HP)} &= \sqrt{[2j - N_k^a]_{q,s^{-1}}}A_k,\\ (J_k^0)^{(HP)} &= -j + N_k^a\end{aligned} \tag{57}$$

(2) The multimode $SU(1,1)_{q,s}$ algebra, (i) Positive discrete case

$$\begin{aligned}(J_k^{(a)})_+^{(HP)} &= s^{-1}A_k^+\sqrt{[2l + N_k^a]_{q,s}},\\ (J_k^{(a)})_-^{(HP)} &= s^{-1}\sqrt{[2l + N_k^a]_{q,s}}A_k,\end{aligned} \tag{58}$$

$$(J_k^{(a)})_0^{(HP)} = l + N_k^a$$

(ii) Negative discrete case

$$\begin{aligned}(J_k^{(b)})_+^{(HP)} &= s\sqrt{[2l + N_k^b]_{q,s^{-1}}}B_k,\\ (J_k^{(b)})_-^{(HP)} &= sB_k^+\sqrt{[2l + N_k^b]_{q,s^{-1}}},\\ (J_k^{(b)})_0^{(HP)} &= -(l + N_k^b)\end{aligned} \tag{59}$$

It is easy to ckeck that Eq. (57) holds Eq. (18), and Eqs. (58) and (59) hold Eqs. (41), (42) and Eqs. (43), (44), respectively.

To sum up, with the help of the multimode q, s-Usui operators, we have obtained the multimode-boson realizations of the multimode $U_{q,s}(SU(2))$ and $U_{q,s}(SU(1,1))$ algebras.

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